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OPTIMAL CONSTRUCTION OF A MASS–SPRING SYSTEM WITH PRESCRIBED MODAL AND SPECTRAL DATA

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The problem of constructing a mass–spring system with prescribed natural frequencies and mode shapes is an overdetermined problem. The independent components of the eigenpairs consist of more constraints than mass and spring values, the free parameters in the problem. The nature of this problem requires the use of non-linear approximation methods. In this paper, two methods of solution, both optimal in some sense, are presented. One method guarantees the global optimal solution with extensive computational effort. The second method evaluates a local optimum in an economical way. The results appearing in this paper may have applications in the design and identification of vibratory systems.

1. INTRODUCTION

Inverse vibration problems associated with the construction of vibratory systems from the known set of desired natural frequencies and mode shapes have many applications in engineering. These include system reconstruction, modification and design. The book by Gladwell [1] introduces the theory of inverse vibration problems. In this paper, we are concerned with the design problem of constructing a physically realizable mass–spring system with prescribed natural frequencies and mode shapes. This problem arises when controlling the maximal deflection of vibratory systems. In reference [2], Zimoch has presented a method for constructing mass and stiffness matrices with prescribed natural frequencies and mode shapes. However, the resulting matrices were not physically realizable in general; i.e., they did not necessarily correspond to real systems with appropriate physical parameters. Ram and Caldwell [3] have shown how to construct a multiple connected mass–spring system from given natural frequencies, and Gladwell and Movahhedy [4] have obtained the set of necessary and sufficient conditions to ensure positive mass and stiffness parameters for the three-degree-of-freedom case. Starek and Inman [5] have analyzed an inverse eigenvalue problem of a non-conservative system. The developed method of solution ensured that the mass, stiffness and damping matrices are real, provided that all eigenvalues of the system are complex. In reference [6] the authors have improved the method to ensure that the matrices are also symmetric, thus enhancing the physical realizability properties of the solution. In reference [7], the method has been further developed to include systems with real eigenvalues associated with overdamped modes.

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Let us now focus on our problem. Consider the symmetric definite generalized eigenvalue problem

$$\mathbf{K}\boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Lambda}, \tag{1}$$

where $\mathbf{K}$ is a positive semi-definite symmetric *stiffness matrix*, $\mathbf{M}$ is a positive definite symmetric *mass matrix*, $\boldsymbol{\Phi}$ is a *mass-normalized modal matrix*, $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, is a *spectral matrix*, and n is the number of *degrees of freedom*.

It is well known that the following bi-orthogonality relations hold:

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}_n, \qquad \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Lambda}. \tag{2, 3}$$

For a multiple connected mass–spring system, the mass matrix $\mathbf{M}$ is real, positive and diagonal. Denote

$$\mathbf{M} = \mathrm{diag}(m_1, m_2, \ldots, m_n), \quad m_i > 0, \qquad m_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n. \tag{4}$$

The stiffness matrix $\mathbf{K} = [k_{ij}]$ is symmetric, and has the following properties:

(a) $\qquad k_{ii} > 0, \qquad i = 1, 2, \ldots, n,$

(b) $\qquad k_{ij} \leqslant 0, \qquad i \neq j, \quad i = 1, 2, \ldots, n; \quad j = 2, 3, \ldots, n;$

(c) $$\sum_{j=1}^{n} k_{ij} \geqslant 0, \qquad i = 1, 2, \ldots, n. \tag{5}$$

In words, $\mathbf{K}$ has positive diagonal elements and non-positive off-diagonal elements, and is weakly diagonally dominant.

Suppose that we want to determine a physically realizable mass–spring system which has a prescribed eigenvalue matrix $\boldsymbol{\Lambda}$ and corresponding mode shapes matrix $\boldsymbol{\Phi}$. If we use the orthogonality equations (2) and (3), we have

$$\mathbf{M} = \boldsymbol{\Phi}^{-\mathrm{T}}\boldsymbol{\Phi}^{-1}, \qquad \mathbf{K} = \boldsymbol{\Phi}^{-\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}. \tag{6, 7}$$

However, in general, this solution would not be physically realizable. Since equations (6) and (7) represent the unique solution to equations (2) and (3), we conclude that generally there is no exact physically realizable solution to this problem. However, we may obtain a physically realizable system with spectral properties that are close to the required data, by solving the following optimization problem.

Problem 1: <u>*Determination of a physically realizable system*.</u> Given sets of desired eigenvalues $\{\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\}$ and corresponding mass-normalized eigenvectors $\{\phi_1^*, \phi_2^*, \ldots, \phi_n^*\}$, denote by

$$\boldsymbol{\Phi}^* = [\phi_1^* | \phi_2^* | \cdots | \phi_n^*] \tag{8}$$

the column partitioning of $\boldsymbol{\Phi}^*$, and let

$$\boldsymbol{\Lambda}^* = \mathrm{diag}(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*). \tag{9}$$

Determine physically realizable $\mathbf{K}$ and $\mathbf{M}$ corresponding to a discrete mass–spring system, with modal and spectral properties $\boldsymbol{\Phi}$ and $\boldsymbol{\Lambda}$ satisfying equation (1), such that the norms $\|\boldsymbol{\Phi}^* - \boldsymbol{\Phi}\|$ and $\|\boldsymbol{\Lambda}^* - \boldsymbol{\Lambda}\|$ are minimized.

We realize that the two problems of determining $\boldsymbol{\Phi}$ and $\boldsymbol{\Lambda}$ can be solved separately. Also note that satisfying equations (2) and (3) is a sufficient condition for equation (1) to hold.

Now consider the problem of determining the optimal mode shape matrix $\boldsymbol{\Phi}$.

Problem 2: <u>*Determination of mode shapes*.</u> Given $\boldsymbol{\Phi}^*$, determine $\boldsymbol{\Phi}$ such that $\mathbf{M} = \boldsymbol{\Phi}^{-\mathrm{T}}\boldsymbol{\Phi}^{-1}$ is a diagonal positive definite matrix, and which minimizes the norm $\|\boldsymbol{\Phi}^* - \boldsymbol{\Phi}\|$.

We analyze this problem in section 2. Once the solution $\mathbf{\Phi}$ is found, we solve the following problem.

Problem 3: *Determination of eigenvalue matrix*. Given $\mathbf{\Lambda}^*$ and $\mathbf{\Phi}$, determine $\mathbf{\Lambda}$ which minimizes the norm $\|\mathbf{\Lambda}^* - \mathbf{\Lambda}\|$, such that $\mathbf{K} = \mathbf{\Phi}^{-\mathrm{T}}\mathbf{\Lambda}\mathbf{\Phi}^{-1}$ satisfies the properties given by equation (5).

We present the global optimal solution to this problem in section 3. Determining the global optimal solution in computationally expensive. We therefore present another, local optimal approximation in section 4. A numerical example demonstrating the algorithms is presented in section 5, and conclusions are drawn in section 6.

2. MODE SHAPE OPTIMIZATION

Let $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $d_i \neq 0$, and let $\mathbf{Q}$ be an orthonormal matrix; that is, $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}_n$. If $\mathbf{\Phi} = \mathbf{D}\mathbf{Q}$, then the mass matrix $\mathbf{M}$ obtained by equation (6) is physically realizable, since

$$\mathbf{M} = \mathbf{\Phi}^{-\mathrm{T}}\mathbf{\Phi}^{-1} = (\mathbf{D}^{-1}\mathbf{Q})(\mathbf{Q}^{\mathrm{T}}\mathbf{D}^{-1}) = \mathbf{D}^{-1}\mathbf{D}^{-1} \tag{10}$$

is a positive definite diagonal matrix. Thus a solution to Problem 2 can be obtained by determining a diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{Q}$, such that

$$\min_{\mathbf{D},\mathbf{Q}} \|\mathbf{\Phi}^* - \mathbf{D}\mathbf{Q}\|. \tag{11}$$

In solving this problem we will make use of the following result. Given two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, the well known *orthogonal Procrustes problem* is to determine an orthonormal matrix $\mathbf{Q}$, such that

$$\min_{\mathbf{Q}} \|\mathbf{A} - \mathbf{B}\mathbf{Q}\|_F. \tag{12}$$

An algorithm for solving this problem is given below (see, e.g., Golub and van Loan [8, p. 582]).

Algorithm 1: *Orthogonal Procrustes problem*

Input: Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$.

Algorithm: (1) Set $\mathbf{C} = \mathbf{B}^{\mathrm{T}}\mathbf{A}$.
(2) Compute the singular value decomposition $\mathbf{C} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$.
(3) Evaluate $\mathbf{Q} = \mathbf{U}\mathbf{V}^{\mathrm{T}}$.

Output: Orthonormal $\mathbf{Q}$, which solves equation (12).

Thus we may choose a diagonal matrix $\mathbf{D}_0$ as an initial guess and obtain an orthonormal $\mathbf{Q}_0$ which minimizes $\|\mathbf{\Phi}^* - \mathbf{D}_0\mathbf{Q}_0\|_F$, by using Algorithm 1. We now show how to obtain a matrix $\mathbf{D}_1$ such that

$$\|\mathbf{\Phi}^* - \mathbf{D}_1\mathbf{Q}_0\|_F \leqslant \|\mathbf{\Phi}^* - \mathbf{D}_0\mathbf{Q}_0\|_F. \tag{13}$$

The Frobenius norm is invariant under orthonormal multiplication. Hence

$$\|\mathbf{\Phi}^* - \mathbf{D}_1\mathbf{Q}_0\|_F = \|\mathbf{\Phi}^*\mathbf{Q}_0^{\mathrm{T}} - \mathbf{D}_1\|_F, \tag{14}$$

Define

$$\mathbf{R} = \mathbf{\Phi}^*\mathbf{Q}_0^{\mathrm{T}}, \tag{15}$$

and denote

$$\varepsilon = \|\mathbf{R} - \mathbf{D}_1\|_F^2. \tag{16}$$

Using the equality

$$\|\mathbf{R} - \mathbf{D}_1\|_F^2 = trace(\mathbf{R}^T\mathbf{R}) + trace(\mathbf{D}_1^T\mathbf{D}_1) - 2\, trace(\mathbf{D}_1^T\mathbf{R}), \tag{17}$$

we find that

$$\varepsilon = trace(\mathbf{R}^T\mathbf{R}) + \sum_{i=1}^{n} [d_{ii}^2 - 2d_{ii}r_{ii}] \tag{18}$$

$$= trace(\mathbf{R}^T\mathbf{R}) + \sum_{i=1}^{n} [d_{ii} - r_{ii}]^2 - \sum_{i=1}^{n} r_{ii}^2, \tag{19}$$

where $\mathbf{D}_1 = \mathrm{diag}(d_{ii})$ and $\mathbf{R} = [r_{ij}]$. Then, from (19), it is clear that ε is minimized when

$$d_{ii} = r_{ii}. \tag{20}$$

Thus, the residual error ε is minimized when the diagonal elements of $\mathbf{D}_1$ are equal to the diagonal elements of $\mathbf{R}$. Having determined a diagonal matrix $\mathbf{D}_1$ satisfying equation (13), we can re-apply Algorithm 1 with $\mathbf{\Phi}^*$ and $\mathbf{D}_1$ as an input and find an orthonormal matrix $\mathbf{Q}_1$ such that

$$\|\mathbf{\Phi}^* - \mathbf{D}_1\mathbf{Q}_1\|_F \leqslant \|\mathbf{\Phi}^* - \mathbf{D}_1\mathbf{Q}_0\|_F. \tag{21}$$

Continuing in this manner iteratively, we obtain an approximation to Problem 2. The following algorithm summarizes this result.

Algorithm 2: Approximate solution to Problem 2

Input: An $n \times n$ modal matrix $\mathbf{\Phi}^*$.

Algorithm: (1) Set initial guess $\mathbf{D}_0$ and a tolerance for convergence $\boldsymbol{\epsilon}$.
- (2) For $i = 0, 1, 2, \ldots$:
  - (a) Evaluate $\mathbf{C} = \mathbf{D}_i^T\mathbf{\Phi}^*$.
  - (b) Compute the singular value decomposition $\mathbf{C} = \mathbf{U\Sigma V}^T$.
  - (c) Evaluate $\mathbf{Q}_i = \mathbf{UV}^T$.
  - (d) Obtain $\mathbf{R} = \mathbf{\Phi}^*\mathbf{Q}_i^T$.
  - (e) $\mathbf{D}_{i+1} = \mathrm{diag}(r_{11}, r_{22}, \ldots, r_{nn})$.
  - (f) Test convergence:
    - (i) Set $N_1 = \|\mathbf{\Phi}^* - \mathbf{D}_i\mathbf{Q}_i\|_F$, $N_2 = \|\mathbf{\Phi}^* - \mathbf{D}_{i+1}\mathbf{Q}_i\|_F$.
    - (ii) If $(N_1 - N_2) \leqslant \boldsymbol{\epsilon}$, go to (3).
- (3) $\mathbf{D} = \mathbf{D}_{i+1}$, $\mathbf{Q} = \mathbf{Q}_i$.

Output: A diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{Q}$ which approximate the solution of equation (11).

It follows from equations (13) and (21) that $\|\mathbf{\Phi}^* - \mathbf{D}_i\mathbf{Q}_i\|_F$ is a monotonic non-increasing function of an iteration index i. Algorithm 2 thus necessarily converges.

3. GLOBAL OPTIMIZATION FOR EIGENVALUES

Using the method described in section 2, we obtain a matrix $\mathbf{\Phi} = \mathbf{DQ}$, which satisfies the physical realisability criteria for $\mathbf{M}$ while minimizing $\|\mathbf{\Phi}^* - \mathbf{DQ}\|_F$. In this section we will use this result to obtain a physically realizable $\mathbf{K}$ which satisfies equation (3) while minimizing $\|\mathbf{\Lambda}^* - \mathbf{\Lambda}\|$.

The physical realizability criteria for the connectivity of $\mathbf{K}$, as described in equation (5), arise from the requirement that the stiffness of all the springs in mass–spring systems must be non-negative. Thus, if we ensure that all the springs have non-negative stiffness, then we necessarily satisfy the conditions of (5).

The stiffness matrix $\mathbf{K}$ may be written in the following form:

$$\mathbf{K} = \sum_{p=0}^{n-1} \sum_{q=p+1}^{n} s_{pq} \mathbf{B}_{pq}, \tag{22}$$

where s_{pq} is the stiffness of the spring connecting mass p to mass q, s_{op} represents the stiffness of the spring which connects mass p to the ground, and $\mathbf{B}_{pq}$ is the matrix describing the spring connection between mass p and mass q:

$$\mathbf{B}_{pq} = [b_{ij}] = \begin{cases} b_{pp} = b_{qq} = 1, & \\ b_{pq} = b_{qp} = -1 & (p \neq q), \\ b_{ij} = 0 & \text{elsewhere.} \end{cases} \tag{23}$$

Substituting equation (22) into equation (3), we obtain

$$\mathbf{\Lambda} = \sum_{p=0}^{n-1} \sum_{q=p+1}^{n} s_{pq} (\mathbf{\Phi}^{\mathrm{T}} \mathbf{B}_{pq} \mathbf{\Phi}). \tag{24}$$

Each of the ijth elements of $\mathbf{\Lambda}$ is thus given by

$$\lambda_{ij} = \sum_{p=0}^{n-1} \sum_{q=p+1}^{n} s_{pq} (\boldsymbol{\phi}_i^{\mathrm{T}} \mathbf{B}_{pq} \boldsymbol{\phi}_j). \tag{25}$$

Let $N = \frac{1}{2}(n^2 + n)$ and construct the vectors

$$\mathbf{y} = (y_1, y_2, y_3, \ldots, y_N)^{\mathrm{T}} = (\lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots, \lambda_{1n}, \lambda_{22}, \lambda_{23}, \ldots, \lambda_{2n}, \lambda_{33}, \ldots, \lambda_{nn})^{\mathrm{T}} \tag{26}$$

and

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_N)^{\mathrm{T}} = (s_{01}, s_{12}, s_{13}, \ldots, s_{1n}, s_{02}, s_{23}, \ldots, s_{2n}, s_{03}, \ldots, s_{nn})^{\mathrm{T}}. \tag{27}$$

Denote

$$\mathbf{F} = [f_{ij}] = \partial y_i / \partial x_j \qquad (i, j = 1, 2, \ldots, N) \tag{28}$$

then all the elements of $\mathbf{F}$ can be evaluated using equation (25). Equation (24) can be written in a vector form:

$$\mathbf{Fx} = \mathbf{y}. \tag{29}$$

In order to satisfy the physical realizability criteria, we require all of the elements of $\mathbf{x}$ to be non-negative. Setting $\mathbf{\Lambda} = \mathbf{\Lambda}^*$ we may determine the vector $\mathbf{y}$ and solve the following *non-negative least squares problem*:

$$\min_{\mathbf{x}} \|\mathbf{F}\mathbf{x} - \mathbf{y}\|_2, \qquad \text{subject to } \mathbf{x} \geqslant 0. \tag{30}$$

An algorithm for the solution of this problem is given in reference [9, p. 161]. (The standard MATLAB function **nnls** solves this problem). Thus the stiffnesses s_{pq} can be obtained from the solution $\mathbf{x}$ of equation (30), via equation (27), which in turn determines matrix $\mathbf{K}$ by equation (22).

The above process gives an optimal solution to the eigenvalue matrix optimization problem, because it is the best positive solution in a least square sense. We note that in order to obtain a solution for the n-degree-of-freedom system, we need to solve an augmented system (30) of dimension N. This is a computational barrier, and an alternative approach is presented in the next section.

4. LOCAL OPTIMIZATION FOR EIGENVALUES

Alternatively, the stiffness matrix $\mathbf{K}$ may be obtained by a local optimization procedure. Setting $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^*$ and multiplying both sides of equation (3) by $\boldsymbol{\Phi}^{-1}$, we have

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K} = \boldsymbol{\Lambda}^*\boldsymbol{\Phi}^{-1}. \tag{31}$$

Denote

$$\mathbf{A} = \boldsymbol{\Lambda}^*\boldsymbol{\Phi}^{-1} \tag{32}$$

and partition $\mathbf{A}$ and $\mathbf{K}$ as follows:

$$\mathbf{A} = [\mathbf{a}_1|\mathbf{a}_2|\mathbf{a}_3| \ldots . |\mathbf{a}_n], \tag{33}$$

$$\mathbf{K} = [\mathbf{k}_1|\mathbf{k}_2|\mathbf{k}_3| \ldots .. |\mathbf{k}_n]. \tag{34}$$

Then, from equation (31), each column of $\mathbf{A}$ is given by

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{k}_j = \mathbf{a}_j \qquad (j = 1, \ldots, n). \tag{35}$$

We now show how to solve equation (35) column by column sequentially. The stiffness matrix $\mathbf{K}$ for a general mass–spring system of order n has the following form

$$\mathbf{K} = \begin{bmatrix} k_{11} & -k_{12} & -k_{13} & -k_{14} & \cdots & -k_{1n} \\ -k_{21} & k_{22} & -k_{23} & -k_{24} & \cdots & -k_{2n} \\ -k_{31} & -k_{32} & k_{33} & -k_{34} & \cdots & -k_{3n} \\ -k_{41} & -k_{42} & -k_{43} & k_{44} & \cdots & -k_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{n1} & -k_{n2} & -k_{n3} & -k_{n4} & \cdots & k_{nn} \end{bmatrix} \tag{36}$$

and physical realizability requires that

(a) $$k_{ij} = k_{ji} \geqslant 0, \qquad \text{for all } 1 \leqslant i, j \leqslant n,$$

(b) $$k_{jj} - \sum_{\substack{i=1 \\ i \neq j}}^{n} k_{ji} \geqslant 0 \qquad (j = 1, 2, \ldots, n). \tag{37}$$

The physical parameters appearing in the first column of **K** can be approximated by solving

$$\min_{\mathbf{k}_1} \|\mathbf{\Phi}^{\mathrm{T}}\mathbf{k}_1 - \mathbf{a}_1\|, \qquad \text{subject to } \mathbf{G}_{(1)}\mathbf{k}_1 \geqslant 0, \tag{38}$$

where

$$\mathbf{G}_{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix}, \qquad \mathbf{G}_{(1)} \in \mathbb{R}^{n \times n}. \tag{39}$$

Then, setting

$$\mathbf{z}_1 = \mathbf{G}_{(1)}\mathbf{k}_1 \qquad \text{and} \qquad \mathbf{E}_{(1)} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{G}_{(1)}^{-1}, \tag{40, 41}$$

we find that equation (38) can be transformed to the standard non-negative least squares form:

$$\min_{\mathbf{z}_1} \|\mathbf{E}_{(1)}\mathbf{z}_1 - \mathbf{a}_1\|, \qquad \text{subject to } \mathbf{z}_1 \geqslant 0. \tag{42}$$

The solution $\mathbf{z}_1$ of equation (42) then determines the physical stiffnesses in $\mathbf{k}_1$, as shown:

$$\mathbf{k}_1 = \mathbf{G}_{(1)}^{-1}\mathbf{z}_1. \tag{43}$$

In a similar manner, the physical parameters appearing in $\mathbf{k}_j$, the jth column of **K**, can be approximated. By the symmetry of **K**, the first $j-1$ elements in the jth step have already been determined in the previous steps. Hence, denoting

$$\hat{\mathbf{k}}_j = \left[-k_{1j}, \ldots, -k_{j-1j}, \sum_{i=1}^{j-1} k_{ij}, 0, 0, \ldots, 0 \right]^{\mathrm{T}}, \tag{44}$$

$$\bar{\mathbf{k}}_j = [0, \ldots, 0, \bar{k}_{jj}, -k_{j+1j}, \ldots, -k_{nj}]^{\mathrm{T}}, \tag{45}$$

we may write

$$\mathbf{k}_j = \hat{\mathbf{k}}_j + \bar{\mathbf{k}}_j, \tag{46}$$

where $\hat{\mathbf{k}}_j$ is known and $\bar{\mathbf{k}}_j$ is to be determined. Substituting equation (46) into equation (35) gives

$$\mathbf{\Phi}^{\mathrm{T}}\hat{\mathbf{k}}_j + \mathbf{\Phi}^{\mathrm{T}}\bar{\mathbf{k}}_j = \mathbf{a}_j. \tag{47}$$

Let $\mathbf{\Phi}$ be partitioned in the form

$$\mathbf{\Phi} = \left[\frac{\mathbf{\Psi}}{\mathbf{\Phi}_{(j)}} \right], \qquad \mathbf{\Phi}_{(j)} \in \mathbb{R}^{(n-j+1) \times n}. \tag{48}$$

Define

$$\mathbf{a}_j^* = \mathbf{a}_j - \mathbf{\Phi}^{\mathrm{T}}\hat{\mathbf{k}}_j, \tag{49}$$

and by truncating the zero elements of the vector $\bar{\mathbf{k}}_j$ in equation (45), set

$$\mathbf{k}_j^* = [\bar{k}_{jj}, -k_{j+1j}, \ldots, -k_{nj}]. \tag{50}$$

Then a non-negative $\mathbf{k}_j^*$ which approximates the solution of equation (47) in least square sense can be obtained by solving

$$\min_{\mathbf{k}_j^*} \|\mathbf{\Phi}_{(j)}^{\mathrm{T}}\mathbf{k}_j^* - \mathbf{a}_j^*\|, \qquad \text{subject to } \mathbf{G}_{(j)}\mathbf{k}_j^* \geqslant 0, \tag{51}$$

where

$$\mathbf{G}_{(j)} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}, \qquad \mathbf{G}_{(j)} \in \mathbb{R}^{(n-j+1)\times(n-j+1)}. \tag{52}$$

Denote

$$\mathbf{z}_j = \mathbf{G}_{(j)}\mathbf{k}_j^* \qquad \text{and} \qquad \mathbf{E}_{(j)} = \mathbf{\Phi}_{(j)}^{\mathrm{T}}\mathbf{G}_{(j)}^{-1}. \tag{53, 54}$$

Then the standard non-negative least square form of equation (51) is given by

$$\min_{\mathbf{z}_j} \|\mathbf{E}_{(j)}\mathbf{z}_j - \mathbf{a}_j^*\|, \qquad \text{subject to } \mathbf{z}_j \geqslant 0. \tag{55}$$

Solving equation (55) for $\mathbf{z}_j$, then $\mathbf{k}_j^*$ can be obtained from

$$\mathbf{k}_j^* = \mathbf{G}_{(j)}^{-1}\mathbf{z}_j. \tag{56}$$

This determines the unknown stiffnesses in the jth column of $\mathbf{K}$. Applying this process for $j = 2, \ldots, n$ evaluates the complete matrix $\mathbf{K}$ in a physically realizable form. The following algorithm summarizes the above process.

Algorithm 3: Approximate solution of Problem 3

Input: A modal matrix $\mathbf{\Phi}$ (as obtained in section 2), and a desired spectral matrix $\mathbf{\Lambda}^*$.

Algorithm: (1) Calculate $\mathbf{A}$ using equation (32) and partition $\mathbf{A}$ as in equation (33). This determines the vectors $\mathbf{a}_j$, $j = 1, 2, \cdots, n$.

(2) Construct the matrix $\mathbf{G}_{(1)}$ as in equation (39).

(3) Determine the matrix $\mathbf{E}_{(1)}$ using equation (41).

(4) Determine the vector $\mathbf{z}_1$ by solving the non-negative least square problem (42).

(5) Obtain $\mathbf{k}_1$ from equation (43). This determines the first row and column of $\mathbf{K} = [k_{ij}]$.

(6) For $j = 2, 3, \ldots, n$:

- (a) Set the vector $\hat{\mathbf{k}}_j$ using equation (44).
- (b) Obtain $\mathbf{\Phi}_{(j)}$ by partitioning $\mathbf{\Phi}$ as in equation (48).
- (c) Determine $\mathbf{a}_j^*$ from equation (49).
- (d) Construct $\mathbf{G}_{(j)}$ as in equation (52) and calculate $\mathbf{E}_{(j)}$ by equation (54).
- (e) Determine $\mathbf{z}_j$ by solving the non-negative least square problem (55).
- (f) Calculate $\mathbf{k}_j^*$ from equation (56).
- (g) Construct vector $\bar{\mathbf{k}}_j$ by augmenting $\mathbf{k}_j^*$ with zero elements as shown in equations (45) and (50).

(h) Obtain $\mathbf{k}_j$ from equation (46). This determines the jth row and column of $\mathbf{K}$, without destroying the symmetry of its first $j-1$ rows and columns.

Output: A physically realizable stiffness matrix $\mathbf{K}$ which approximates the solution of Problem 3 in the local optimization sense.

The computational expense of this process is approximately equal to solving n times a non-negative least square problem of dimensions $n, (n-1), \ldots, 1$. This is more efficient than solving an augmented system of dimension N.

5. NUMERICAL EXAMPLE

The local optimization solution obtained by Algorithm 3 is not the optimal solution in the global sense, such as described in section 3. It is shown in this section, by means of a numerical example, that the quality of solution is not greatly affected.

Suppose that the desired dynamic properties, $\mathbf{\Lambda}^*$ and $\mathbf{\Phi}^*$, for a five-degree-of-freedom mass–spring system are

$$\mathbf{\Lambda}^* = \mathrm{diag}(50, 100, 200, 400, 800)$$

and

$$\mathbf{\Phi}^* = \begin{bmatrix} 0{\cdot}1 & -0{\cdot}1 & 0{\cdot}2 & -0{\cdot}4 & 0{\cdot}1 \\ 0{\cdot}1 & 0{\cdot}1 & 0{\cdot}2 & 0{\cdot}1 & 0{\cdot}3 \\ 0{\cdot}1 & -0{\cdot}1 & 0{\cdot}3 & 0{\cdot}2 & -0{\cdot}4 \\ 0{\cdot}1 & -0{\cdot}3 & -0{\cdot}1 & -0{\cdot}1 & -0{\cdot}1 \\ 0{\cdot}3 & 0{\cdot}2 & -0{\cdot}1 & 0{\cdot}1 & 0{\cdot}1 \end{bmatrix}.$$

We wish to determine physically realizable $\mathbf{M}$ and $\mathbf{K}$ which have dynamic characteristics as close as possible to the above data.

We note that there is no exact solution for these data since, by equations (6) and (7),

$$\mathbf{M} = \mathbf{\Phi}^{*-\mathrm{T}}\mathbf{\Phi}^{*-1} = \begin{bmatrix} 6{\cdot}6406 & -4{\cdot}5515 & 1{\cdot}0830 & -4{\cdot}7646 & 2{\cdot}7310 \\ -4{\cdot}5515 & 13{\cdot}5005 & -0{\cdot}3195 & 8{\cdot}6019 & -4{\cdot}9737 \\ 1{\cdot}0830 & -0{\cdot}3195 & 3{\cdot}5646 & -1{\cdot}1215 & 1{\cdot}0213 \\ -4{\cdot}7646 & 8{\cdot}6019 & -1{\cdot}1215 & 14{\cdot}5886 & -1{\cdot}5877 \\ 2{\cdot}7310 & -4{\cdot}9737 & 1{\cdot}0213 & -1{\cdot}5877 & 8{\cdot}9672 \end{bmatrix}$$

and

$$\mathbf{K} = \mathbf{\Phi}^{*-\mathrm{T}}\mathbf{\Lambda}^*\mathbf{\Phi}^{*-1} = \begin{bmatrix} 2216{\cdot}1 & -2100{\cdot}0 & 230{\cdot}0 & -1446{\cdot}8 & 476{\cdot}9 \\ -2100{\cdot}0 & 6358{\cdot}5 & -1{\cdot}6181 & 2763{\cdot}1 & -1762{\cdot}7 \\ 230{\cdot}0 & -1618{\cdot}1 & 1559{\cdot}2 & -915{\cdot}4 & 352{\cdot}6 \\ -1446{\cdot}8 & 2763{\cdot}1 & -915{\cdot}4 & 2324{\cdot}0 & -699{\cdot}3 \\ 476{\cdot}9 & -1762{\cdot}7 & 352{\cdot}6 & -699{\cdot}3 & 945{\cdot}7 \end{bmatrix}$$

which consists of a non-physically realizable solution. We now show how to determine an optimal solution.

Applying Algorithm 2, we obtain a diagonal matrix **D** and an orthonormal matrix **Q**, such that $\mathbf{\Phi} = \mathbf{DQ}$ is given by

$$\mathbf{\Phi} = \begin{bmatrix} 0{\cdot}1232 & -0{\cdot}0333 & 0{\cdot}2107 & -0{\cdot}3988 & 0{\cdot}0414 \\ 0{\cdot}0578 & -0{\cdot}0363 & 0{\cdot}1922 & 0{\cdot}1503 & 0{\cdot}2680 \\ 0{\cdot}1337 & -0{\cdot}0707 & 0{\cdot}3292 & 0{\cdot}1820 & -0{\cdot}3767 \\ 0{\cdot}1186 & -0{\cdot}2856 & -0{\cdot}1045 & 0{\cdot}0061 & 0{\cdot}0073 \\ 0{\cdot}3233 & 0{\cdot}1725 & -0{\cdot}1021 & 0{\cdot}0324 & 0{\cdot}0088 \end{bmatrix}$$

and $\|\mathbf{\Phi}^* - \mathbf{\Phi}\|_F$ is minimized.

Substituting $\mathbf{\Phi}$ in equations (6) and (7), we obtain

$$\mathbf{M}' = \mathrm{diag}(4{\cdot}5152,\ 7{\cdot}3516,\ 3{\cdot}2650,\ 9{\cdot}3757,\ 6{\cdot}8583)$$

and

$$\mathbf{K}' = \begin{bmatrix} 1523{\cdot}9 & -216{\cdot}2 & -392{\cdot}0 & -146{\cdot}0 & -240{\cdot}6 \\ -216{\cdot}2 & 4009{\cdot}4 & -1356{\cdot}4 & -48{\cdot}7 & 10{\cdot}5 \\ -392{\cdot}0 & -1356{\cdot}4 & 1597{\cdot}1 & -178{\cdot}5 & -135{\cdot}7 \\ -146{\cdot}0 & -48{\cdot}7 & -178{\cdot}5 & 976{\cdot}0 & -47{\cdot}9 \\ -240{\cdot}9 & 10{\cdot}5 & -135{\cdot}7 & -47{\cdot}9 & 506{\cdot}5 \end{bmatrix}.$$

The mass matrix $\mathbf{M}'$ is now physically realizable, whereas the stiffness matrix $\mathbf{K}'$ is not realizable. Therefore, setting $\mathbf{\Lambda} = \mathbf{\Lambda}^*$ and applying the global optimization procedure described in section 3, we obtain the following realizable stiffness matrix:

$$\mathbf{K}'' = \begin{bmatrix} 1512{\cdot}0 & -227{\cdot}2 & -337{\cdot}7 & -144{\cdot}3 & -245{\cdot}4 \\ -227{\cdot}2 & 4012{\cdot}4 & -1277{\cdot}9 & -41{\cdot}7 & 0 \\ -337{\cdot}7 & -1277{\cdot}9 & 1690{\cdot}1 & -37{\cdot}6 & -36{\cdot}9 \\ -144{\cdot}3 & -41{\cdot}7 & -37{\cdot}6 & 939{\cdot}8 & -69{\cdot}1 \\ -245{\cdot}4 & 0 & -36{\cdot}9 & -69{\cdot}1 & 454{\cdot}2 \end{bmatrix}.$$

Setting $\mathbf{M} = \mathbf{M}'$ and $\mathbf{K} = \mathbf{K}''$, we have a realizable mass–spring system with the following modal data:

$$\mathbf{\Lambda}' = \mathrm{diag}(52{\cdot}8,\ 101{\cdot}2,\ 214{\cdot}3,\ 401{\cdot}3,\ 795{\cdot}2),$$

$$\mathbf{\Phi}' = \begin{bmatrix} 0{\cdot}0982 & -0{\cdot}0265 & 0{\cdot}2488 & -0{\cdot}3848 & 0{\cdot}0335 \\ 0{\cdot}0267 & -0{\cdot}0165 & 0{\cdot}1995 & 0{\cdot}1600 & 0{\cdot}2639 \\ 0{\cdot}0553 & -0{\cdot}0273 & 0{\cdot}3379 & 0{\cdot}2010 & -0{\cdot}3846 \\ 0{\cdot}0939 & -0{\cdot}3087 & -0{\cdot}0488 & 0{\cdot}0137 & 0{\cdot}0002 \\ 0{\cdot}3538 & 0{\cdot}1203 & -0{\cdot}0691 & 0{\cdot}0375 & 0{\cdot}0012 \end{bmatrix}.$$

This compares reasonably well with the desired properties $\mathbf{\Lambda}^*$ and $\mathbf{\Phi}^*$.

This solution is computationally expensive. Applying Algorithm 3 to the above example, we obtain a physically realizable stiffness matrix

$$\mathbf{K}''' = \begin{bmatrix} 1523{\cdot}9 & -216{\cdot}2 & -392{\cdot}0 & -146{\cdot}0 & -240{\cdot}6 \\ -216{\cdot}2 & 4009{\cdot}3 & -1356{\cdot}4 & -48{\cdot}7 & 0 \\ -392{\cdot}0 & -1356{\cdot}4 & 1748{\cdot}4 & 0 & 0 \\ -146{\cdot}0 & -48{\cdot}7 & 0 & 976{\cdot}0 & -47{\cdot}9 \\ -240{\cdot}6 & 0 & 0 & -47{\cdot}9 & 506{\cdot}5 \end{bmatrix}.$$

With $\mathbf{M} = \mathbf{M}'$ and $\mathbf{K} = \mathbf{K}'''$, the modal data of the system are

$$\mathbf{\Lambda}'' = \mathrm{diag}(62{\cdot}6,\ 104{\cdot}0,\ 200{\cdot}5,\ 407{\cdot}6,\ 821{\cdot}7),$$

$$\mathbf{\Phi}'' = \begin{bmatrix} 0{\cdot}0985 & -0{\cdot}0314 & 0{\cdot}2479 & -0{\cdot}3838 & 0{\cdot}0454 \\ 0{\cdot}0251 & -0{\cdot}0174 & 0{\cdot}2013 & 0{\cdot}1683 & 0{\cdot}2574 \\ 0{\cdot}0471 & -0{\cdot}0255 & 0{\cdot}3384 & 0{\cdot}1863 & -0{\cdot}3927 \\ 0{\cdot}0841 & -0{\cdot}3116 & -0{\cdot}0474 & 0{\cdot}0161 & -0{\cdot}0028 \\ 0{\cdot}3578 & 0{\cdot}1087 & -0{\cdot}0661 & 0{\cdot}0400 & -0{\cdot}0021 \end{bmatrix}.$$

We note that the global optimal solution is slightly better than the local one. However, they both lead to essentially similar systems. In Table 1 are shown the cosines of the angles between the desired mode shapes and the modes of the physical systems which have been obtained. Let θ be the angle between two eigenvectors. Then $\cos\theta = 1$ indicates identical eigenvectors.

TABLE 1

Cosines of angles between the desired mode shapes and their approximations

	Mode number, j				
	1	2	3	4	5
Cos (ϕ_j^*, ϕ_j)	0·9885	0·9210	0·9988	0·9586	0·9580
Cos (ϕ_j^*, ϕ_j')	0·9648	0·9015	0·9852	0·9541	0·9558
Cos (ϕ_j^*, ϕ_j'')	0·9587	0·8948	0·9845	0·9506	0·9571

We sought mass-normalized eigenvectors: hence the amplitude ratio between the desired mode shapes and their approximation is also of interest. These amplitude ratios are given in Table 2. The results in Tables 1 and 2 present a good agreement between the desired mode shapes and the modes obtained.

TABLE 2

Amplitude ratios between the desired mode shapes and their approximations

	Mode number, j				
	1	2	3	4	5
$\|\phi_j\|/\|\phi_j^*\|$	1·0918	0·8617	1·0541	0·9688	0·8773
$\|\phi_j'\|/\|\phi_j^*\|$	1·0649	0·8346	1·0835	0·9683	0·8838
$\|\phi_j''\|/\|\phi_j^*\|$	1·0657	0·8323	1·0835	0·9605	0·8915

6. CONCLUSIONS

The problem of constructing a mass–spring system with prescribed eigenvalues and mode shapes has been addressed. This is a non-linear approximation problem, since the number of constraints, the eigendata, is larger than the number of free parameters, the number of masses and springs in the system.

It is shown that the problem of determining the mass and stiffness matrices can be solved separately. First, an optimal set of mode shapes associated with a physically realizable mass matrix is obtained. This is done by a convergent iterative algorithm. Then a physically realizable stiffness matrix is determined using the optimal mode shapes obtained in the previous stage.

Two methods of obtaining a physically realizable stiffness matrix have been suggested. One method determines a global optimal solution in a least square sense. This method involves non-linear optimization of large matrices of order N for a problem with n degrees of freedom. The other method breaks the problem into n sub-problems of small dimensions, and determines a local optimal solution for each sub-problem. The result is a computationally economical method of solution. It is shown through a numerical example that both methods lead to similar solutions.

The results presented in this paper may be applied in designing physically realizable systems with prescribed spectral constrains, and in identifying realizable systems from modal test data.

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